EUROPEAN ORGANIZATION FOR NUCLEAR RESEARCH

# Delamotte Approximation

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**Abstract**

We solve the anharmonic potential with an approximate, non perturbative method of B. Delamotte. The numerical problem is analyzed and the analytical solution is obtained in Jacobi Elliptical functions. We show, that the Fourier expansion of $Sd(u||m)$ the Jacobi function is the Delamotte approximation. We analyze the problems of the method when there are several equilibrium points. As in Duffing equation.
For the limit cycles we use the simple nonlinear oscillation as a model to study the convergency of the method. The problems about the uniqueness of the approximation are studied.
Two theorems are proved that give a practical criteria when to use the method for Hamiltonian systems, and dissipative systems with a unique asymptotically stable limit cycle.

# 1 Introduction

Dr. B. Delamotte of the Laboratory of High Energy Physics from Paris University, published [1] what we call the Delamotte ansatz. The ansatz is a non perturbative method for solving second order differential equations. The method is good for periodic solutions and the solution it's obtained independent of the values of the "perturbation parameter". In some cases, as the Fourier spectrum of the solution becomes broad the convergence of the method is slow. In other words as the first Fourier coefficients are small, the slower the convergence is. The method is a uniform approximation, that could be used if the solutions are periodic of period T and $C^2[0,T]$. With a coincidence with the real solution at the initial point in position and velocity and a uniform approximation in position, velocity and acceleration, the method is indeed a very good approximation in very low orders.

Delamotte ansatz is the following[2]. Let call x(t) the solution of the second order differential autonomous equation:

$$f(x(t), x'(t), x''(t)) = 0 \ \mathbf{I-1}$$

subject to the initial conditions

$$x(t_0) = x_0, \qquad x'(t_0) = x'_0 \ \mathbf{I-2}$$

The principle of the method is to replace eq. I-1 and I-2 by a linear differential equation with a explicit time dependent right-hand side. In the language of classical mechanics an external force, that forces the harmonic potential to follow the trajectory x(t). This force always exists and is given by:

$$x''(t) + \omega^2 x(t) = F(t) \ \mathbf{I-3}$$

where $\omega$ is a free parameter. Note that F depends on the particular differential equation. The principle of the method is to make the ansatz:

$$F_{ans}(t) = x''_{ans} + \omega^2 x_{ans}(t) \cdot \mathbf{I-4}$$

if $\delta x$ is the difference between $x$ and $x_{ans}$

$$x(t) = x_{ans}(t) + \delta x(t) \ \mathbf{I-5}$$

and $x_{ans}$ is close enough to x if:

$$|\delta x| \ll |x_{ans}|; \qquad |\delta x'| \ll |x'_{ans}|; \qquad |\delta x''| \ll |x''_{ans}| \ \mathbf{I-6}$$

In practice, $x_{ans}$ is expanded on a basis of functions, in our case as a Fourier sum:

$$x_{ans}(t) = \sum_{k=0}^{N} x_k sin(k\omega t) + y_k cos(k\omega t) \mathbf{I-7}$$

[1] Phys. Rev. Lett. Volume 70 Number 22 of may 31, 1993 page 3361

[2] The equations I-n are from Delamotte article.

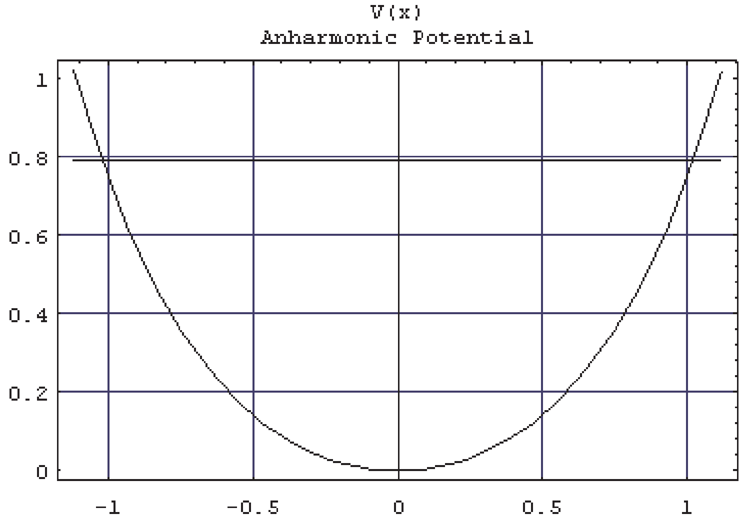


Figure 1: The anharmonic potential and total energy

## 2 The anharmonic potential

Let consider the anharmonic oscillator of equation:

$$x'' = -\omega_0^2 x - g x^3 \ \mathbf{I-8}$$

$$\delta x'' = -\omega_0^2 \delta x - (\omega_0^2 - \omega^2) x_{ans} - g(x_{ans} + \delta x)^3 - F_{ans} \ \mathbf{I-9}$$

using I-6 we have:

$$(\omega_0^2 - \omega^2) x_{ans} + g x_{ans}^3 + F_{ans} \sim 0 \mathbf{I-10}$$

Let V(x) be the anharmonic potential:

$$V(x) = \frac{1}{2} x^2 + \frac{g}{4} x^4 \ \mathbf{1}$$

the corresponding differential equation is:

$$\frac{d^2 x}{dt^2} + x + g x^3 = 0 \ \cdot \mathbf{2}$$

where we did the mass m, the spring constant k and the frequency $w_0^2$ equal to 1. The energy conservation could be written as:

$$\frac{1}{2} v^2 + \frac{1}{2} x^2 + \frac{g}{4} x^4 = \frac{1}{2} v_0^2 \ \cdot \mathbf{3}$$

The turning point has equation:

$$g x^4 + 2x^2 - 2v_0^2 = 0 \ \mathbf{4}$$

The solution of equation (4) is:

$$x_r = \pm \left\{ \frac{[1 + 2g v_0^2]^{1/2} - 1}{g} \right\}^{1/2} \ \mathbf{5}$$

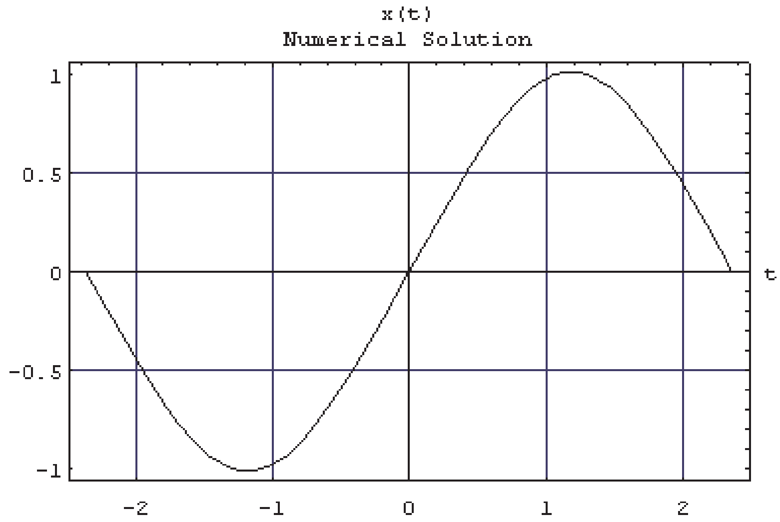


Figure 2: The solution of the anharmonic potential

### 2.1 The First Integral

The energy or first integral of equation (2) is:

$$\int_0^x \frac{dz}{\sqrt{2v_0^2 - 2z^2 - gz^4}} = \int_0^t \frac{dz}{\sqrt{2}} \cdot \cdot \mathbf{6}$$

Equation (6) could be integrated[3)]. In order to use the formula we have to complete the square.

$$2v_0^2 - 2x^2 - gx^4 = g\left\{\sqrt{\frac{2v_0^2}{g} + \frac{1}{g^2}} - (\frac{1}{g} + x^2)\right\}\left\{\sqrt{\frac{2v_0^2}{g} + \frac{1}{g^2}} + (\frac{1}{g} + x^2)\right\} \cdot \mathbf{7}$$

This means that:

$$a^2 = \sqrt{\frac{2v_0^2 g + 1}{g^2}} + \frac{1}{g} \; \mathbf{8}$$

$$b^2 = \sqrt{\frac{2v_0^2 g + 1}{g^2}} - \frac{1}{g} \; \mathbf{9}$$

The solution which we call Se(t) is:

$$Se(t) = \frac{v_0}{[1 + 2v_0^2 g]^{1/4}} Sd\left([1 + 2v_0^2 g]^{1/4} t \| \frac{1}{2}(1 - \frac{g^2}{\sqrt{1 + 2v_0^2 g}}\right) \; \mathbf{10 - A}$$

This is the analytical solution of the $x^4$ potential for a particle with $x(0) = 0$ and $v(0) = v_0$. For figures 1,2 and 3 we have chosen the following parameters:
For equation I-8 $w_0 = 1$ and $g = 1$
For the initial conditions I-2 $x_0 = 0$ and $v_0 = 1.256$

[3)] Formula 17.4.51 page 596 from the Abramowitz and Stegun; Handbook of Mathematical Functions.

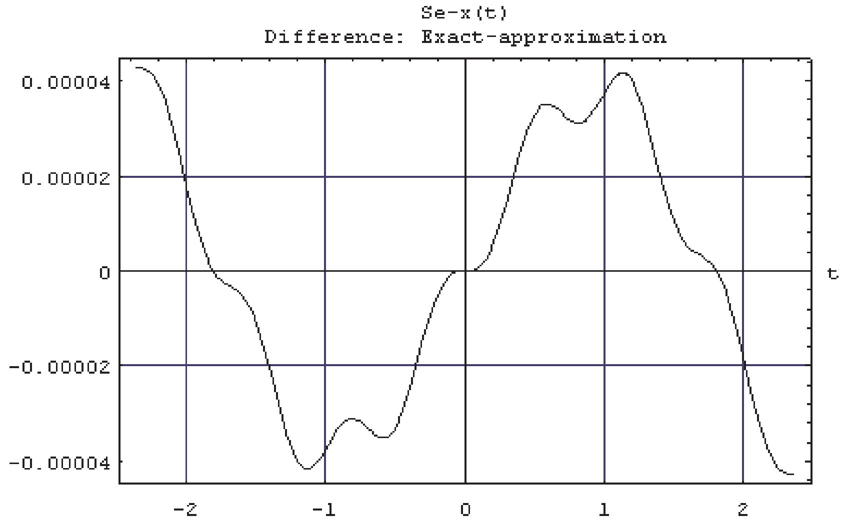


Figure 3: Difference between analytical and Delamotte approximation

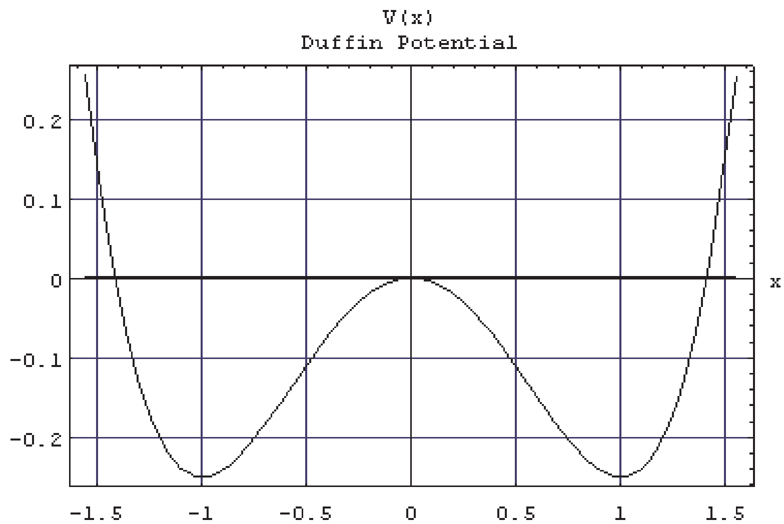


Figure 4: Duffing potential

## 3 The Generalization to Duffing Equation

The equation for the anharmonic oscillator that we have analyzed could be generalized by the introduction of a parameter $\lambda$, the equation could be written as[4]:

$$\frac{d^2w}{dt^2} + \lambda w + g w^3 = 0 \quad \mathbf{11}$$

where now $\lambda \; \epsilon \; [-1, 1]$ is a parameter for the sign of linear force, besides g, that is the parameter of the strength of the perturbation.

For figures 4,5 and 6 we have chosen:
For equation 11 $\lambda = -1$, $g = 1$
For the initial conditions I-2 $x_0 = 0$ and $v_0 = -0.0467$

[4] J. Hale and Kocak, Dynamics and Bifurcations,Springer Verlag, 1991

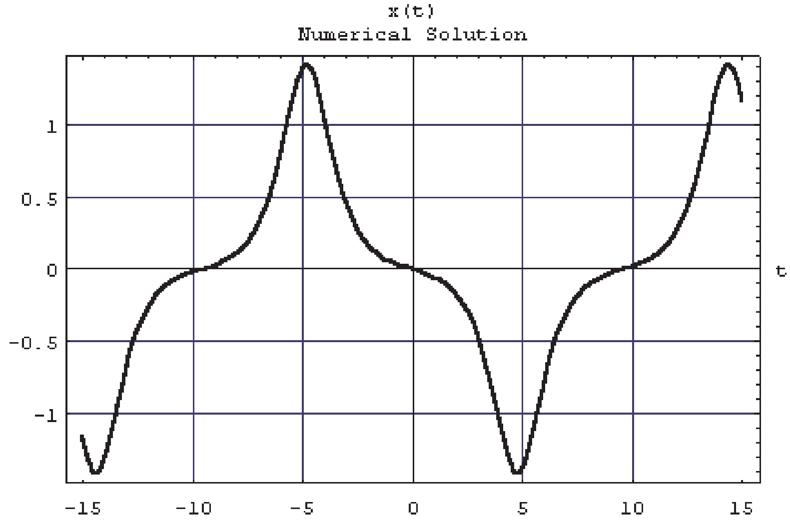


Figure 5: Duffing numerical solution

The analytical solution of equation 11 is a modification of equation 10-A:

$$Se(t) = \frac{v_0 \lambda^{1/4}}{[\lambda + 2v_0^2 g]^{1/4}} Sd\left( [\lambda(\lambda + 2v_0^2 g)]^{1/4} t \| \frac{1}{2}(1 - \frac{g^2}{\lambda^{3/2}\sqrt{\lambda + 2v_0^2 g}} \right) \cdot \mathbf{10-B}$$

Equation 11 could be written as a linear system with the substitution:

$$x = w \qquad y = \frac{dw}{dt} \cdot \cdot \mathbf{12}$$

Then

$$\frac{dx}{dt} = y = f(x, y)$$

$$\frac{dy}{dt} = -\lambda x - g x^3 = g(x, y) \cdot \mathbf{13}$$

We now look for the equilibrium points given by the equations:

$$f(x, y) = 0 \qquad g(x, y) = 0 \cdot \cdot \mathbf{14}$$

with solutions:

$$(0, 0) \qquad (\pm \sqrt{-\frac{\lambda}{g}}, 0) \cdot \cdot \mathbf{15}$$

If $\lambda \geq 0$ there is only 1 equilibrium point the origin (0,0), if $\lambda \leq 0$ there are 3 equilibrium points given by equation 15.

We need to relate the stability of the equilibrium points with the application of the Delamotte ansatz, we study the stability of the 3 equilibrium points through the linearization, let's write the Floquet matrix:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Using definition 13 for f and g we obtain:

$$m = \begin{bmatrix} 0 & 1 \\ -\lambda - 3gx^2 & 0 \end{bmatrix} \cdot \cdot \mathbf{16}$$

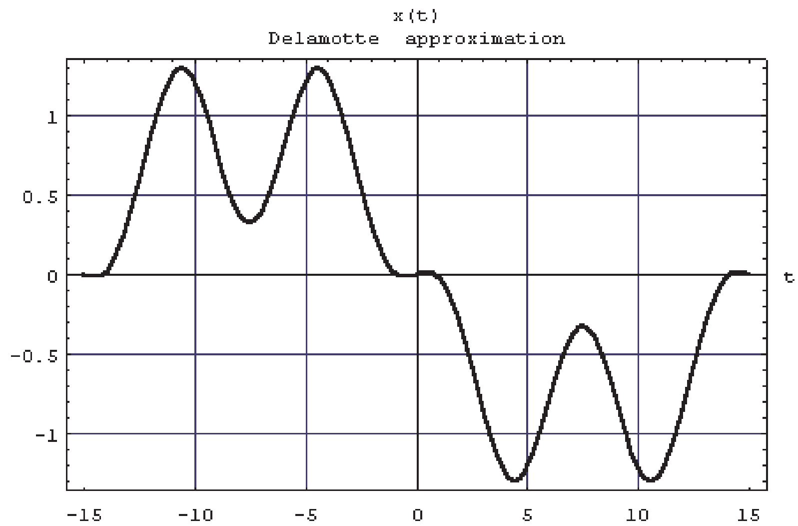


Figure 6: Delamotte approximation to Duffing

In order to study the stability of each equilibrium point we need the eigenvalues of m in equation 16 for each point.

$$m_{\pm} = \pm\sqrt{-\lambda - 3gx^2} \mathbf{17}$$

now we have 3 cases to analyse:
1) (0,0) $m_{\pm} = \pm\sqrt{-\lambda}$ if $\lambda < 0$, $m_{\pm} = \pm\sqrt{|\lambda|}$ then it has 2 eigenvalues different from 0 and opposite signs and (0,0) is a saddle.
2) In the case $\lambda < 0$ and $x = \pm\sqrt{\frac{-\lambda}{g}}$ the eigenvalues are $m_{\pm} = \pm i\sqrt{2|\lambda|}$ the result is a center we can not apply the theorem to the linearization but the potential theory say it's a minimum or stable equilibrium point[5]. For this case the Delamotte ansatz do not work, there are several real roots for the values of $\omega$ and $x_1$ etc.

## 4 Numerical Conclusions

Mathematica [6] numerical calculations show(but not prove):
1)Convergency of Delamotte ansatz to the Fourier coefficients in the anharmonic potential.
2)In the case $\lambda < 0$, the method works for a big total energy $E$.
3)But as $E \to 0$ or $E < 0$ there are inconsistencies not only related to Delamotte ansatz but to the analytical method of solution. In the case of Delamotte ansatz several real roots appear in (I-10), in the analytical case you have to choose a proper analytical extension for equation (10-B).

The real problem is near $E \to 0$, because once you have chosen a specific "vacuum" (minimum of the potential) the solution could be found in the neighborhood. Some measure of the speed of convergency is the Fourier spectrum or the absolute value of the Fast Fourier Transform of the solution figure 7 shows the FFT for Duffing in the case $E \to 0$.

In the following section we state some conclusions of our numerical work under the generic name of Delamotte theorems.

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[5] Lemma 14.2 Kocak
[6] Wolfram Research, Windows Version 2.1

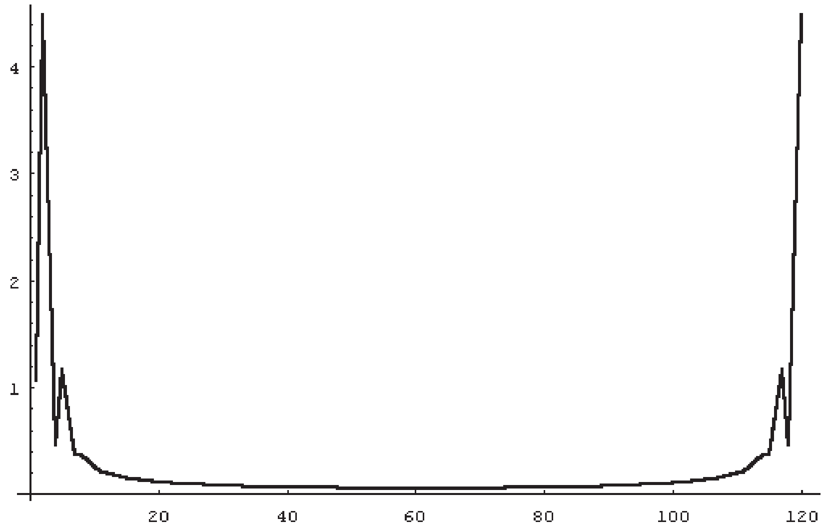


Figure 7: FFT of Duffing Solution

## 5 Delamotte Theorems

As we have seen the Delamotte ansatz gives a very good approximation for some kinds of differential equations. The big problems are to characterize the specific class of differential equations and to prove the convergency to the Fourier expansion of the solution. We write some theorems, which give specifics implementations of Delamotte ansatz.

We divide the theorems in 2 cases:The Hamiltonian systems with a unique minimum of the potential. As we have seen these systems oscillate and the Fourier series expansion is natural [7].Dissipative systems with a unique asymptotically stable limit cycle.

### 5.1 Hamiltonian Systems

A Hamiltonian system is one with a constant Hamiltonian, the energy, for this system a potential V(x) exists and the force is written as:

$$F = -\frac{\partial V}{\partial x} \cdot \cdot \mathbf{18}$$

then the equation I-1 is written as:

$$\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0 \qquad x(0) = x_0, \qquad x'(0) = v_0 \cdot \mathbf{19}$$

and the energy E:

$$E = \frac{1}{2}(\frac{dx}{dt})^2 + V(x) \cdot \cdot \mathbf{20}$$

is a constant.

**Theorem # 1 Let be $V(x)\epsilon C^2[a,b]$ and $x(t)\epsilon C^2[0,T]$ where T is the period of the solution and the differential equation:**

$$\frac{d^2x}{dt^2} + \frac{\partial V}{\partial x} = 0 \qquad x(0) = x_0, \qquad x'(0) = v_0 \cdot \mathbf{19}$$

[7] Ecuaciones diferenciales y cálculo variacional, L. Elsgoltz, Ediciones de Cultura Popular, 1975,page 146

**if $V(x) = E$ has only two real roots a,b and only one local minimum of V(x), $a < x_{min} < b$; then, exist:**
**A driving force; where w is a free parameter**

$$F_{ans}(t) = x''_{ans} + \omega^2 x_{ans}(t) \mathbf{I-4}$$

**An expansion,**

$$x_{ans}(t) = \sum_{k=0}^{N} x_k sin(k\omega t) + y_k cos(k\omega t) \mathbf{I-7}$$

**And if $x_{ans}$ is close enough,**

$$|\delta x| \ll |x_{ans}|; \qquad |\delta x'| \ll |x'_{ans}|; \qquad |\delta x''| \ll |x''_{ans}| \cdot \mathbf{I-6}$$

**I-7 is the Fourier series of the solution $x(t)$.** *Proof. Assume that equation 19 satisfy the conditions of the theorem of existence, uniqueness and smoothness*[8] *and that the interval [0,T] is contained on the maximal interval of existence of the equation.*

We define the Delamotte topology in the sense that if $g, f \epsilon C^2[0,T]$

$$||g|| = \sup_{t\epsilon[0,T]} |g(t)|$$

There is a technical term for Delamotte topology is called the $C^2$ topology[9]:

$$||f-g||_2 = \sup_{t\epsilon[0,T]} \{||D^i f(x) - D^i g(x)||_{i\le 2}\}$$

Is evident that $x_{ans}$ as defined in I-7 is in $C^2[0,T]$ in other words Delamotte convergence:

$$||g - f|| \ll ||f||$$

is convergence in the uniform norm:

$$\sup_{t\epsilon[0,T]} ||g(t) - f(t)|| < \epsilon \sup_{t\epsilon[0,T]} |f(t)|$$

By Delamotte topology we understand the $(x, x', x'') \to (x_a, x'_a, x''_a)$ if

$$|\delta x| \ll |x_{ans}|; \qquad |\delta x'| \ll |x'_{ans}|; \qquad |\delta x''| \ll |x''_{ans}|$$

in the sense:

$$\sup_{t\epsilon[0,T]} |x(t) - x_{ans}(t)| < \epsilon_1; \qquad \sup_{t\epsilon[0,T]} |x'(t) - x'_{ans}(t)| < \epsilon_2; \qquad \sup_{t\epsilon[0,T]} |x''(t) - x''_{ans}(t)| < \epsilon_3$$

Delamotte topology is stronger than the normal notion of neighborhood in the sense that systems that have similar values of x(t) and v(t) are alike. In Delamotte topology is necessary to have similar forces to be alike.

Delamotte topology could be though in geometric terms if we think of the phase portrait of the differential equation as the evolution curve of the initial conditions, in this sense the evolution curve and the corresponding Delamotte approximation are not only close in the phase space but they must have the same curvature, in this sense there is a

[8] As in Kocak, page 542
[9] Kocak, page 390

high degree of contact between both curves, note that in the plane this is the maximum degree as there is no torsion in a plane.

As we know the uniform norm dominates the $L^2$ semi norm [10] and together with the theorem[11] about the uniqueness of the Fourier expansion, the Delamotte expansion is the Fourier expansion.

See that the other conditions about the derivatives are necessary to obtain the coefficients and the condition about the roots are in order to have a fast numerical convergence.

For example the second difference (the last term of equation I-6) essentially say that the convergence is faster that $\frac{C}{n^2}$. The limitation in V(x) is not essential for the convergence but gives to the method a practical condition when the convergence is fast.

## 6 Limit Cycles

### 6.1 Lienard differential equation

In this section we use the simple nonlinear oscillations[12] in order to study the value of Delamotte approximation to differential equations with a unique asymptotically stable limit cycle.[13]

The Smith equations are worth to mention because they have a formal solution in terms of elementary functions, the associated limit cycles are algebraic curves and for some values of the parameters the differential equation is similar to Van der Pol equation.

We understand as Lienard equation the differential equation:

$$x''(t) + x'(t)f(x) + g(x) = 0 \cdot \mathbf{21}$$

with the following conditions:

f(x) is continuous and even, and $f(0) < 0$

g(x) is continuous, satisfies the Lipschitz condition, and $xg(x) > 0$, for $x \neq 0$

$F(x) \to \pm\infty$ as $x \to \pm\infty$, respectively, where $F(x) = \int_0^x f(u)du$, and $F(x)$ has a single positive zero at $x = a$, while for $x \geq a$, $F(x)$ increases monotonically. Then the periodic orbit is hyperbolic and asymptotically stable.

We make the Smith choice:

$$f(x) = (n+2)bx^n - 2a \qquad g(x) = x[c + (bx^n - a)^2] \cdot \mathbf{22}$$

and a,b,$c = \omega^2$ and n are constants. The general solution is:

$$x(t) = \frac{cos(p + \omega t)}{\left\{qe^{-nat} + nbe^{-nat}\int_0^t e^{na\theta}cos^n(p + \omega\theta)d\theta\right\}^{1/n}} \cdot \mathbf{23}$$

We shall study the special case when a=2, b=1, c=1 and n=2, in this case equation 21 is similar to Van der Pol equation:

$$x'' + 4x'[x^2 - 1] + x + x(x^2 - 2)^2 = 0\mathbf{25}$$

The general solution to this equation is:

$$x(t) = \frac{cos(p + t)}{\left\{qe^{-4t} + \frac{1}{4} + \frac{1}{10}[2cos(2(p+t)) + sin(2(p+t))]\right\}^{1/2}} \cdot \mathbf{26}$$

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[10] S. Lang. Analysis I page 104

[11] S. Lang page 235 theorem 9

[12] Described by R. Smith, Journal London Math Soc. 36(1961)33-34

[13] Kocak, page 382

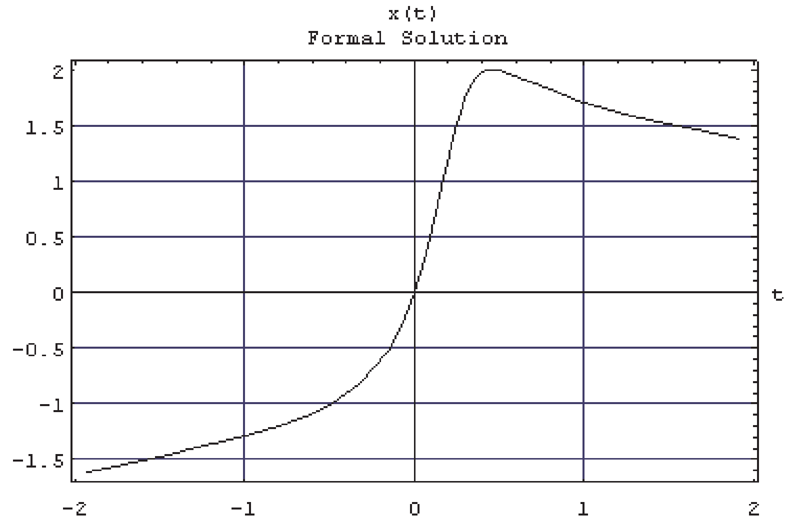


Figure 8: Solution to Smith equation

where p and q are arbitrary constants. When q=0 the solution is periodic, and is easy to see that the period is $T = 2\pi$ in this case we wrote equation 26 as:

$$x(t) = \frac{cos(p+t)}{\sqrt{\frac{1}{4} + \frac{1}{10}[2cos(2(p+t)) + sin(2(p+t))]}} \cdot \mathbf{27}$$

from equation 27 we obtain the special initial conditions for the limit cycle in the case $p = \pm\frac{\pi}{2}$.This is the case when initially the particle is in the origin and has an initial velocity $v_0$, since q and p are fixed $v_0 = \mp 2\sqrt{5}$ in other words the special initial conditions for the limit cycle we choose are:

$$x(0) = x_0 = 0 \qquad x'(0) = v_0 = \mp 2\sqrt{5} \cdot \mathbf{28}$$

this solution has period $T = 2\pi$ and amplitude $x_{max} = 2.0$ .

The graphic of equation 27 is figure 8 and figure 9 the Delamotte low order approximation.

In this case the corresponding Fourier coefficients are $c_n$ :
[2.6660,0.3817,0.5284,0.1772,0.2075,0.0584, 0.0705,0.0,0.0116,-0.0219,-0.0096......] where:

$$c_n = \frac{2}{\pi}\int_0^{\pi} x(t)sin(nt)dt \cdot \mathbf{28}$$

we see that the convergency is rather slow, using the relation for the velocity implied in the Fourier expansion:

$$v_n = \sum_1^{\infty} nc_n \cdot \mathbf{29}$$

since the sum of the first 11 terms give 7.4907 instead of 4.4721 and the first negative term appears for n=9.

### 6.2 Delamotte method to Smith equations

Applying the external force equation I-3 to Smith equation 21 we obtain:

$$\delta x'' = -x'_{ans}f(x_{ans} + \delta x) - g(x_{ans} + \delta x) + \omega^2 x_{ans} - F_{ans} \cdot \mathbf{30}$$

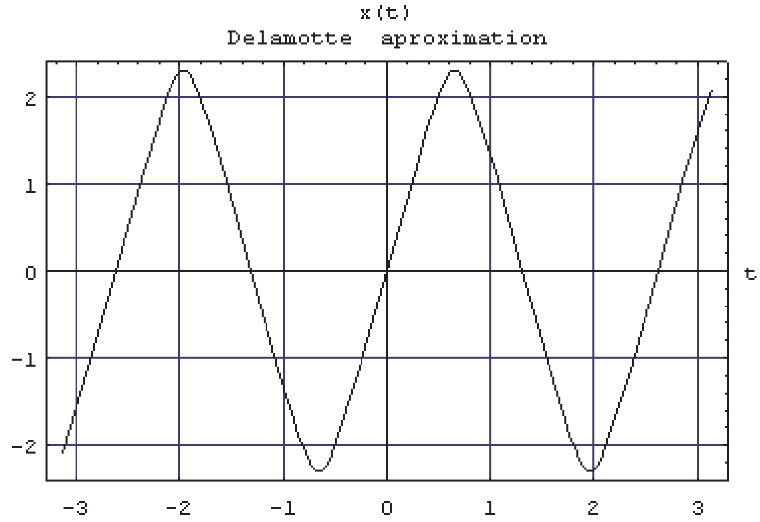


Figure 9: Delamotte approximation to Smith equation

which gives the correct answer for the Van der Pol equation and in our case:

$$-4x'_{ans}(x^2_{ans}-1)+(\omega^2-1)x_{ans}-x_{ans}(x^2_{ans}-2)^2-F_{ans}\sim 0 \cdot \mathbf{31}$$

The class of systems to which we have consider the Delamotte method is called dissipative systems. A system of differential equations is dissipative if whatever the initial condition, there exist some $t_0$ that for $t > t_0$, x(t) the solution is contained in a bounded subset.

The characteristic of dissipative systems with a unique cycle limit is that for a special set of initial conditions there is a periodic orbit, however in general there are several initial conditions that give the same limit cycle. As an example see equation 26 with q=0 and several values of p. On the other hand Delamotte method does not give the quantity of information as easily for all initial conditions, often to obtain the amplitude of the limit cycle is already a problem in the solution of the differential equation, [14) ] Delamotte method is very fast given the correct amplitude if the initial conditions are expressed as $x(0) = x_0,\ v(0) = 0$.

For example if Lienard equation 21 has a static solution $g(x) = 0$ as in Smith equation 25 $x = 0$, Delamotte approximation is:

$$x_{ans}(t)=\sum_{k=1}^{N} x_k sin(k\omega t)+y_k cos(k\omega t) \cdot \mathbf{I-7}$$

In lowest order of approximation:

$$x(t)\simeq y_1 cos(wt)\simeq 2cos(t)$$

the phase portrait of this approximation is a ellipse of equation:

$$\frac{x^2}{2^2}+\frac{v^2}{2^2w^2}=1$$

and area $4\pi w$. Figure 10 shows the phase portrait of Smith equation 25 the ellipse is Delamotte approximation.

[14)] C.C. Chou ASME Page 496 june 1975

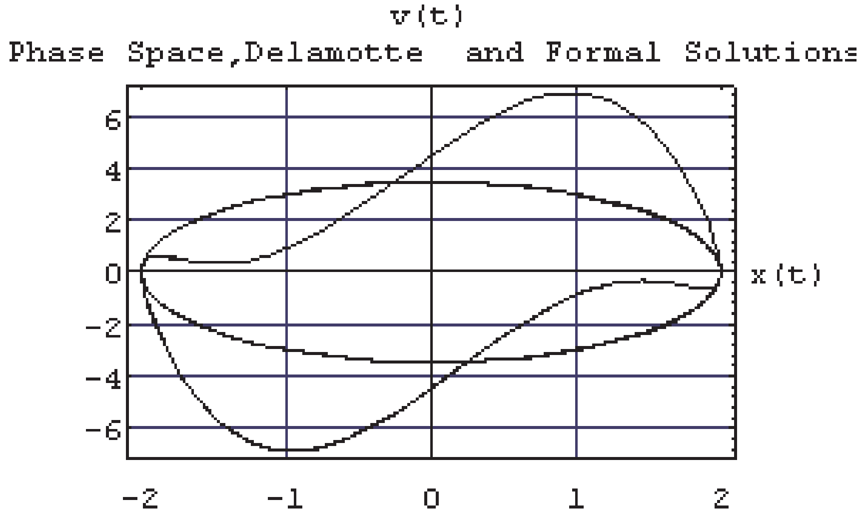


Figure 10: Phase portrait of Smith equation, the ellipse is Delamotte Solution

**Theorem # 2 For a dissipative differential equation, like Lienard**

$$x''(t) + x'(t)f(x) + g(x) = 0 \cdot \mathbf{21}$$

**with a unique asymptotically stable limit cycle, and the set of initial conditions** $x(0) = x_0$ **and** $v(0) = 0$ **Delamotte approximation is the Fourier expansion of the periodic orbit.** *Proof. The convergency is the same as in theorem 1, note here the importance of the asymptotical stability of the orbit for the existence of the sup norm of the solution, we have avoided the unstable case even if it's possible that Delamotte approximation converge to the limit cycle, but would not be near the solutions that spiral out of the limit cycle.*

## 7 Conclusions

Delamotte criteria of convergency or topology is enough for guaranty the $L^2$ convergency to the Fourier series of the solution. Theorem #1 and #2 give practical criteria for a fast convergency of Delamotte method.